From the morphism between QFTs to topological Wick rotation II

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June 16, 2026, Croucher Summer Course: Emergent Symmetry, Chinese University of Hong Kong

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References: K.-Zheng:1705.01087, 1905.04924, 1912.01760,

The principle of 'bulk=Z(boundary)' applies to an n+1D topological order C^{n+1} with a gapped boundary Kitaev-K.1104.5047,K.:1307.8244 and also to that with a gapless boundary A^n. K.-Wen-Zheng:1502.01690, 1702.00673



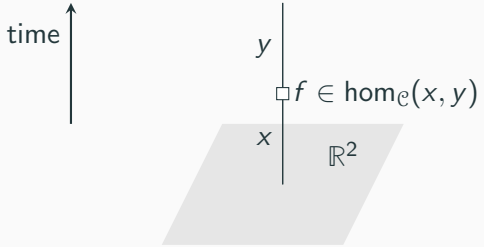
1. When $n = 2$, an anomaly-free 2+1D topological order C^{2+1} can be characterized by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is the modular tensor category (MTC) of anyons and c is the chiral central charge. We should be able to find the categorical characterization of a gapless boundary A^n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$.
2. Since $\mathfrak{Z}_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$, we can view $? \simeq \sqrt{\mathcal{C}}$. I would like to remind you that the taking-square-root problems in history $\sqrt{2}, \sqrt{-1}, \sqrt{\Delta}$ are all very important. Moreover, I believe that boundary $= \sqrt{\text{bulk}}$ should not be very different from gauge theory $= \sqrt{\text{gravity}}$.

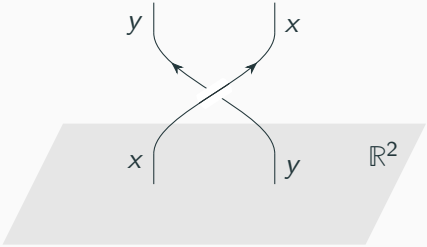
A few comments on finding solutions to the equation $\mathfrak{Z}_1(?) = \mathcal{C}$ when $\mathcal{C}$ is not the Drinfeld center of a fusion category. In this case, the boundaries of $(\mathcal{C}, c)$ have to be gapless.

1. In mathematics, this problem was proposed in 1990s in order to extend the Reshetikhin-Turaev TQFT from a 1-2-3 theory to a 0-1-2-3 theory (i.e., a fully extended TQFT). However, from the mathematical point of view, it is very hard to guess what kind of mathematical structure this ? is.
2. Another "common" reaction to this problem is that since $\mathcal{C}$ is a MTC and we knew that the boundary ? should be a "chiral CFT", which is just a VOA for most working physicists, then this equation $\mathfrak{Z}_1(?) = \mathcal{C}$ does not make sense. Something is wrong. Either "$\mathfrak{Z}_1(?) = \mathcal{C}$" or "? is a VOA" is wrong.
3. Once we accept 'bulk=Z(boundary)' as a guiding principle, we see that ? must come from macroscopic observables living on the 1+1D worldsheet of a gapless boundary of the 2+1D topological order $(\mathcal{C}, c)$. In other words, the mother nature automatically provides the solutions to the equation.

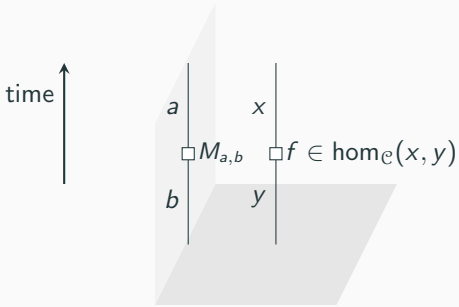
Observables on 1+1D worldsheet of a gapless boundary

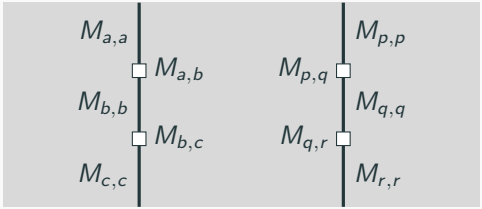
An anomaly-free 2+1D topological order C^{2+1} can characterized by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is the modular tensor category (MTC) of anyons and c is the chiral central charge.





Consider observables on the 1+1D worldsheet of a gapless boundary of the 2+1D topological order $(\mathcal{C}, c)$.

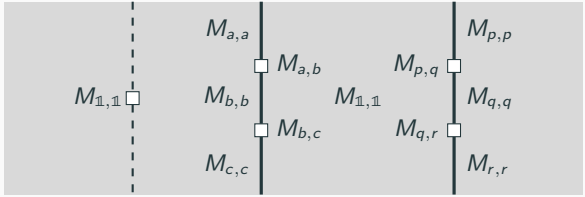




(1) $\mathcal{S} = \{a, b, c, \cdots\}$ is the set of labels of defect lines;

(2) $M_{a,b}$ is the spaces of fields living on the 0D defect junction; in particular, it means that the space of fields living on the defect line label by 'a' is just $M_{a,a}$;

If the boundary is chiral,
then $M_{\mathbb{1},\mathbb{1}} = \{\phi(z)\}$ form a VOA.
If the boundary is non-chiral,
$M_{\mathbb{1},\mathbb{1}}$ is a full field algebra.



(3) $\mathbb{1} \in \mathcal{S}$ labels the trivial defect line. The fields in $M_{\mathbb{1},\mathbb{1}}$ live in the 2-cell. We assume that the boundary is chiral. The chiral fields in $M_{\mathbb{1},\mathbb{1}} = \{\phi(z)\}$ form a structure of vertex operator algebra (VOA).

- $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-1}$;
- OPE = Associativity:

$$\phi(z_1)\psi(z_2) \sim \sum_{k<N_{\psi,\phi}} \frac{(\psi_k\phi)(z_2)}{(z_1-z_2)^{k+1}} + \cdots \quad \Longleftrightarrow \quad Y_{\mathbb{1},\mathbb{1}}(\phi,z_1)Y_{\mathbb{1},\mathbb{1}}(\psi,z_2)\varphi = Y_{\mathbb{1},\mathbb{1}}(Y_{\mathbb{1},\mathbb{1}}(\phi,z_1-z_2)\psi,z_2)\varphi.$$

- Commutativity: $\phi(z_1)\psi(z_2) \sim \psi(z_2)\phi(z_1)$;
- a sub-VOA $\langle T\rangle \subset M_{\mathbb{1},\mathbb{1}}$, $T(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, $[L(m),L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}$.

(4) For a non-trivial defect line $a \neq \mathbb{1}$, $M_{a,a} = \{\phi(r)\}$ forms an open-string VOA. Huang-K.:math/0308248

- $\phi(r) = \sum_{n\in\mathbb{R}} \phi_n r^{-n-1}$; (Since the fields in $M_{a,a}$ are restricted to the TDL, there is no monodromy issue.)
- OPE = Associativity:

$$\phi(z_1)\psi(z_2) \sim \sum_{k<N_{\psi,\phi}} \frac{(\psi_k\phi)(z_2)}{(z_1-z_2)^{k+1}} + \cdots \quad \Longleftrightarrow \quad Y_{a,a}(\phi,z_1)Y_{a,a}(\psi,z_2)\varphi \sim Y_{a,a}(Y_{a,a}(\phi,z_1-z_2)\psi,z_2)\varphi.$$

- Non-Commutativity: $\phi(z_1)\psi(z_2) \nsim \psi(z_2)\phi(z_1)$;
- a sub-VOA $\langle T_{a,a}(r)\rangle \subset M_{a,a}$,

$$T_{a,a}(r) = \sum_{n\in\mathbb{Z}} L(n) r^{-n-2}, \qquad [L(m),L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}.$$

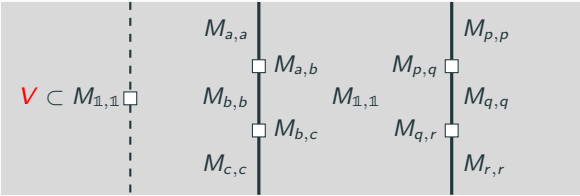
(5) The most general OPE is

$$\begin{array}{ccc} M_{b,c} \otimes_{\mathbb{C}} M_{a,b} & \xrightarrow{Y_{a,b,c}(-,r)-} & M_{a,c} \\ \phi \otimes_{\mathbb{C}} \psi & \mapsto & Y_{a,b,c}(\phi, r)\psi \end{array}$$

(6) Associativity of OPE: $M_{c,d} \otimes_{\mathbb{C}} M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \to M_{a,d}$

$$Y_{a,c,d}(\varphi, r_1) Y_{a,b,c}(\phi, r_2)\psi = Y_{a,c,d}(Y_{b,c,d}(\varphi, r_1 - r_2)\phi, r_2)\psi, \qquad \textit{for } r_1 > r_2 > r_1 - r_2 > 0.$$

Remark: $\left(\mathcal{S} = \{a, b, c, \cdots\},\ \{M_{a,b}\}_{a,b\in\mathcal{S}},\ \{M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{Y_{a,b,c}} M_{a,c}\}_{a,b,c\in\mathcal{S}}\right)$ is almost but not yet a categorical structure. What is missing is the identity morphisms.

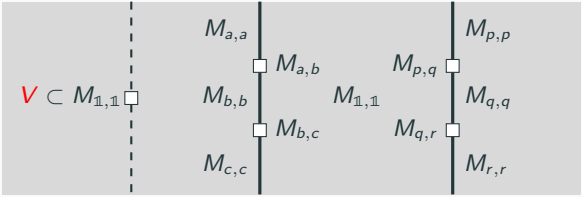


(7) Now we consider only topological defect lines (TDL). By definition, a TDL is a defect that is transparent to the sub-VOA $\langle T(z)\rangle \subset M_{\mathbb{1},\mathbb{1}}$, i.e., there is a path-independent VOA-map $\langle T(z)\rangle \xrightarrow{\simeq} \langle T_{a,a}(r)\rangle \hookrightarrow M_{a,a}$.

(8) A V-symmetric TDL is a TDL that is transparent to the generic sub-VOA V (i.e., $\langle T(z)\rangle \subset V \subset M_{\mathbb{1},\mathbb{1}}$). From now on, we assume that all the TDL's are V-symmetric. This freedom of choice of V suggests that V might be a defining data of the gapless boundary.

- By "transparent to V", we mean that there is a canonical (path independent) and well-defined VOA embedding $V \hookrightarrow M_{a,a}$, which is automatically a morphism in Mod_V.

(9) $M_{a,b}$ is necessarily a V-module equipped with a 2-dimensional V-action, a notion which is called an E_2-V-module, i.e., $M_{a,b} \in \mathrm{Mod}_V^{\mathrm{E}_2} = \mathrm{Mod}_V$. At this stage, it is unnecessary but helpful to understand the situation by assuming V to be rational, i.e. Mod_V is a MTC. Moore-Seiberg:1989, Huang:math/0502533

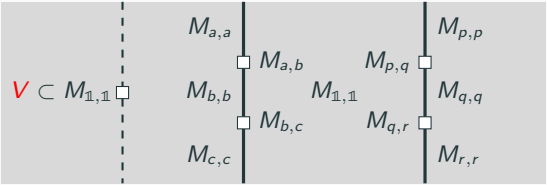


(10) The compatibility between the OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{Y_{a,b,c}} M_{a,c}$ and the V-action means that,

- for $v \in V$, $w \in M_{b,c}, w' \in M_{a,b}$,

$$Y_{M_{a,c}}(v, z_1)Y_{a,b,c}(w, r_2)w' \quad \sim \quad Y_{a,b,c}(Y_{M_{a,b}}(w, z_1 - r_2), r_2)w' \quad \sim \quad Y_{a,b,c}(w, r_2)Y_{M_{a,b}}(v, z_1)w'$$

- This compatibility condition is precisely the defining property for the vertex operator $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{Y_{a,b,c}} M_{a,c}$ to be an intertwining operator of V.
- By Huang-Lepowksy's theory of the tensor product of V-modules, this intertwining operator $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{Y_{a,b,c}} M_{a,c}$ is equivalent to a morphism in $M_{b,c} \otimes_V M_{a,b} \xrightarrow{\circ_{a,b,c}} M_{a,c}$ in Mod_V.



(11) The observables on the 1+1D worldsheet form the following set of data:

$$\Big(\mathcal{S}=\{a,b,c,\cdots\},\ \ \{M_{a,b}\in\mathcal{B}\}_{a,b\in\mathcal{S}},\ \ \{V\hookrightarrow M_{a,a}\}_{a\in\mathcal{S}},\ \ \{M_{b,c}\otimes_V M_{a,b}\xrightarrow{\circ_{a,b,c}} M_{a,c}\}_{a,b,c\in\mathcal{S}}\Big),$$

which, by setting $\mathcal{B}:=\mathrm{Mod}_V$, is precisely a $\mathcal{B}$-enriched category, denoted by ${}^{\mathcal{B}}\mathcal{S}$, such that $\hom_{\mathrm{Mod}_V{}^{\mathcal{B}}\mathcal{S}}(a,b)=M_{a,b}$.

(12) Let $\otimes$ be the horizontal fusion of TDLs. It upgrades ${}^{\mathcal{B}}\mathcal{S}$ to a $\mathcal{B}$-enriched fusion category.

Remark: It turns out that all correlation functions and the OPE among defect fields can be recovered from a triple $(V,{}^{\mathcal{B}}\mathcal{S})$ for $\mathcal{B}=\mathrm{Mod}_V$. Huang:math/0303049,math/0502533, Fuchs-Runkel-Schweigert:2001-2006, Huang-Kirillov-Lepowsky:1406.3420, K.:math/0612255, K-.Runkel:0807.3356, Davydov-K.-Runkel:1307.5956

Definition

Let $\mathcal{B}$ a monoidal category. A $\mathcal{B}$-enriched category ${}^{\mathcal{B}}\mathcal{S}$ consists of the following data:

(1) a set of object $\mathrm{ob}({}^{\mathcal{B}}\mathcal{S})$; (If $x \in \mathrm{ob}({}^{\mathcal{B}}\mathcal{S})$, we also write $x \in {}^{\mathcal{B}}\mathcal{S}$.)

(2) an object $\hom_{{}^{\mathcal{B}}\mathcal{S}}(x, y)$ in $\mathcal{B}$ for every pair $x, y \in {}^{\mathcal{B}}\mathcal{S}$;

(3) a distinguished morphism $1_x : \mathbb{1}_{\mathcal{B}} \to \hom_{{}^{\mathcal{B}}\mathcal{S}}(x, x)$ in $\mathcal{B}$ for $x \in {}^{\mathcal{B}}\mathcal{S}$;

(4) a composition morphism $\hom_{{}^{\mathcal{B}}\mathcal{S}}(y, z) \otimes \hom_{{}^{\mathcal{B}}\mathcal{S}}(x, y) \xrightarrow{\circ_{x,y,z}} \hom_{{}^{\mathcal{B}}\mathcal{S}}(x, z)$ for $x, y, z \in {}^{\mathcal{B}}\mathcal{S}$;

such that the composition is associative and 1_x plays the role of identity morphism.

Example: An ordinary category is a category enriched in the category $\mathcal{S}et$ of sets. A fusion category is a category enriched in the category Vec of finite dimensional vector spaces.

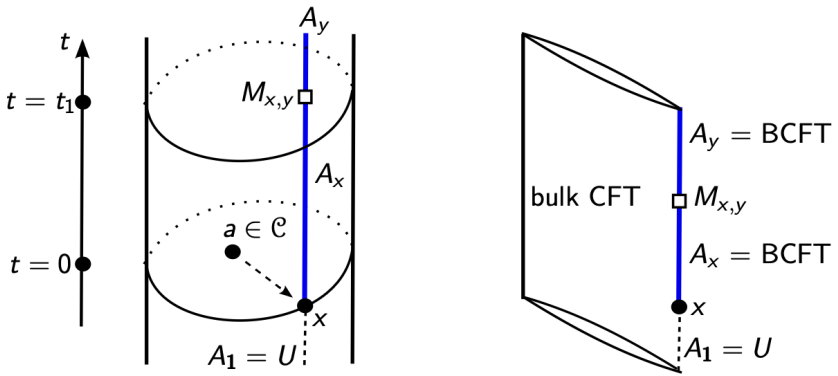
Remark: We define a new category $\mathcal{S}$ by setting $\mathrm{ob}(\mathcal{S}) := \mathrm{ob}({}^{\mathcal{B}}\mathcal{S})$ and $\hom_{\mathcal{S}}(x, y) := \hom_{\mathcal{B}}(\mathbb{1}_{\mathcal{B}}, \hom_{{}^{\mathcal{B}}\mathcal{S}}(x, y))$. This category $\mathcal{S}$ is called the underlying category of ${}^{\mathcal{B}}\mathcal{S}$.

Mathematical characterization of a gapless boundary of a 2+1D topological order

- **Physics helps math**: Mother nature tells you that the mathematical structure of ? in the equation $\mathcal{C} = \mathfrak{Z}_1(?)$ should be a $\mathcal{B}$-enriched fusion category, where $\mathcal{B}$ is a MTC. It is a new type of mathematical structure obtained by analyzing the observables on the 1+1D worldsheet of the gapless boundary without considering the bulk $(\mathcal{C}, c)$ at all.
- **Math helps physics**: This analysis does not tell you the solution of the equation $\mathcal{C} = \mathfrak{Z}_1(?)$ directly. However, it tells you what kind of mathematical structure ? should be. In order to determine ? precisely, we need a mathematical theory of enriched monoidal categories, including the theory of the monoidal center of an enriched monoidal category.
  Once we have developed this mathematical theory. We can find the solutions to the equation $\mathcal{C} = \mathfrak{Z}_1(?)$ within the fusion categories enriched certain modular tensor categories. This solution will further provide a precise mathematical characterization of a gapless boundary of $(\mathcal{C}, c)$.

Now I will skip a lot of technical details and give the results directly. However, I will list the previous known results and new results that are needed for our conclusion.

- A mathematical theory of rational VOA and its representations. Huang:math/0303049,math/0502533
- The classification results of rational CFTs, including (anomaly-free) modular-invariant bulk CFTs, boundaries CFTs and domain wall CFTs. Fuchs-Runkel-Schweigert:2001-2006, K-.Runkel:0807.3356
- The reformulation of all (anomaly-free) modular-invariant bulk CFTs, boundaries CFTs, domain wall CFTs and defect junctions between domain walls in terms of internal homs and enriched categories. Davydov-K.-Runkel:1307.5956
- A mathematical theory of enriched monoidal categories Morrison-Penneys:1701.00567, K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121 and their monoidal centers K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121



Anomaly-free principle: A 1+1D boundary-bulk conformal field theory realized by a 1d lattice Hamiltonian model with boundaries should satisfy the mathematical axioms of a boundary-bulk (or open-closed) CFT of all genera, including modular invariance, Cardy condition, etc.

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

The 'rational' gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, \mathcal{S})$ or $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where

1. *for a chiral gapless boundary, V is a rational VOA of central charge c; Huang:math/0502533*
   for a non-chiral gapless boundary, V is a rational full field algebra ($c_L - c_R = c$) K.-Huang:math/0511328;
   for a gapped boundary, $V = \mathbb{C}$. Kitaev-K.:1104.5047
2. *$\mathcal{S}$ is a fusion category equipped with a braided equivalence $\phi : \mathcal{C} \boxtimes \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathfrak{Z}_1(\mathcal{S})$.*

Moreover, the restriction $\phi : \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})} = \overline{\mathrm{Mod}_V}$ is a braided equivalence, which determines a $\mathcal{B}$-action on $\mathcal{S}$ and a $\mathcal{B}$-enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction.
Lindner:1981,Morrison-Penneys:1701.00567 *More precisely, we have $\hom_{{}^{\mathcal{B}}\mathcal{S}}(x, y) = M_{x,y} = [x, y] \in \mathcal{B}$ for $x, y \in \mathcal{S}$, where $[x, y]$ is called internal hom and is defined by the following natural isomorphisms*

$$\hom_{\mathcal{S}}(b \triangleright x, y) \simeq \hom_{\mathcal{B}}(b, [x, y]), \qquad \text{for } b \in \mathcal{B}, x, y \in \mathcal{S}.$$

Definition (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

Let $\mathcal{C}^\sharp$ be a monoidal category enriched over $\mathcal{B}$. A *half-braiding* for an object $x \in \mathcal{C}^\sharp$ is an enriched natural isomorphism $b_x : x \otimes - \to - \otimes x$ between enriched endo-functors of $\mathcal{C}^\sharp$ such that it defines a half-braiding in the underlying monoidal category $\mathcal{C}$. The Drinfeld center of $\mathcal{C}^\sharp$ is a category $Z(\mathcal{C}^\sharp)$ enriched over $\mathcal{B}$ defined as follows:

- an object is a pair (x, b_x), where $x \in \mathcal{C}^\sharp$ and b_x is a half-braiding for x;
- $\hom_{Z(\mathcal{C}^\sharp)}((x, b_x), (y, b_y))$ is the intersection of the equalizers of the diagrams $\hom_{\mathcal{C}^\sharp}(x, y) \rightrightarrows \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y)$ depicted below for all $z \in \mathcal{C}^\sharp$

$$\begin{array}{ccc} \hom_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\otimes \circ (\mathrm{id}_z \otimes \mathrm{id})} & \hom_{\mathcal{C}^\sharp}(z \otimes x, z \otimes y) \\ \Big\downarrow{\scriptstyle \otimes \circ (\mathrm{id} \otimes \mathrm{id}_z)} & & \Big\downarrow{\scriptstyle - \circ b_{x,z}} \\ \hom_{\mathcal{C}^\sharp}(x \otimes z, y \otimes z) & \xrightarrow{b_{y,z} \circ -} & \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y); \end{array}$$

- the composition law $\circ$ is induced from that of $\mathcal{C}^\sharp$.

Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.

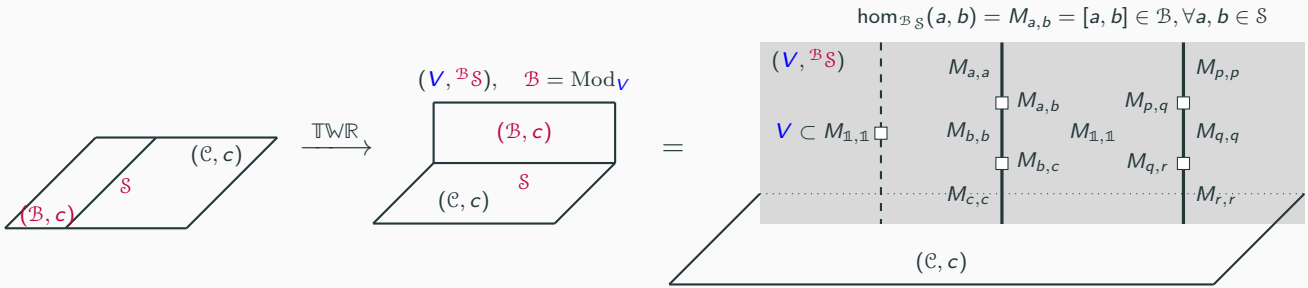
Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a pair* $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$*, where*

1. $\mathsf{X}_{\mathrm{lqs}} = (V, \phi)$ *is called local quantum symmetry (containing dynamical information);*
2. $\mathsf{X}_{\mathrm{top}} = {}^{\mathcal{B}}\mathcal{S}$ *is called topological skeleton (recall* $\mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})}$*).*

Moreover, $\mathsf{X}_{\mathrm{top}}$ *can be obtained by topological Wick rotation and* $\mathfrak{Z}_1(\mathsf{X}_{\mathrm{top}}) \simeq \mathcal{C}$*.*





When $(\mathcal{C}, c)$ is trivial, i.e., $\mathcal{B} = \mathfrak{Z}_1(\mathcal{S})$, we obtain a holographic duality between a 3D theory and a 2D theory. In this case, $\mathcal{B} = \mathfrak{Z}_1(\mathcal{S})$ is what later called "SymTO/SymTFT". We can also see very clearly the famous slogan "symmetry is a topological order in 1-higher dimension".

Examples of gapless boundaries

Examples of gapped and gapless boundaries:

1. For a non-chiral 2+1D topological order $(\mathfrak{Z}(\mathcal{A}), 0)$, where $\mathcal{A}$ is a fusion category, the pair $(\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{A} = \mathcal{A})$, where $\mathrm{Mod}_{\mathbb{C}} = \mathrm{Vec}$, defines a gapped boundary.
2. For the E_8 invertible 2+1D topological order $(\mathcal{C}, c) = (\mathrm{Vec}, 8)$, the pair $(V_{E_8}, {}^{\mathrm{Vec}}\mathrm{Vec})$, where $\mathrm{Mod}_{V_{E_8}} = \mathrm{Vec}$, defines a non-trivial gapless boundary.
3. Let V be a rational VOA and $\mathcal{C} = \mathrm{Mod}_V$ is MTC.
   - The pair $(V, {}^{\mathcal{C}}\mathcal{C})$ defines a gapless boundary of $(\mathcal{C}, c)$ and $\mathfrak{Z}({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$.
   - The triple $(V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C})$ defines an anomaly-free 1+1D CFT, where $\mathrm{Mod}_{V\otimes_{\mathbb{C}}\overline{V}} = \mathcal{C} \boxtimes \overline{\mathcal{C}}$, and

$$M_{\mathbb{1}_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}} = [\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}] = \oplus_{i\in \mathrm{Irr}(\mathcal{C})}\ i \boxtimes i^*;$$
$$M_{a,b} = [a, b] = \oplus_{i\in \mathrm{Irr}(\mathcal{C})}\ (i \otimes b \otimes a^*) \boxtimes i^*$$

   Moreover, we have $\mathfrak{Z}_1({}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C}) = \mathrm{Vec}$.

Examples of chiral gapless edge:

Consider a conformal embedding $V \subsetneq A$ of rational VOA's with the same central charge c. For example:

$$su(m)_n \times su(n)_m \subset su(mn)_1, \quad sp(2m)_n \times sp(2n)_m \subset so(4mn)_1,$$
$$so(m)_n \times so(n)_m \subset so(mn)_1, \quad so(m)_4 \times su(2)_m \subset sp(2m)_1, \cdots.$$



In this case, we have $M_{\mathbb{1},\mathbb{1}} = A \in \mathrm{Mod}_V$, i.e. $V \subsetneq M_{\mathbb{1},\mathbb{1}} = A$.

Examples of non-chiral gapless edges:

- Three modular tensor categories (MTC):
  1. $\mathsf{Is} := \mathrm{Mod}_{V_{\mathsf{Is}}}$ where V_{Is} is the Ising VOA with the central charge $c = \frac{1}{2}$. It has three simple objects $\mathbb{1}, \psi, \sigma$ (i.e. $\mathbb{1} = V_{\mathsf{Is}}$) and the following fusion rules:

$$\psi \otimes \psi = \mathbb{1}, \quad \psi \otimes \sigma = \sigma, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$

  2. $\mathfrak{Z}_1(\mathsf{Is}) \simeq \mathsf{Is} \boxtimes \overline{\mathsf{Is}}$. It has 9 simple objects: $\mathbb{1} \boxtimes \mathbb{1}, \mathbb{1} \boxtimes \psi, \mathbb{1} \boxtimes \sigma, \psi \boxtimes \mathbb{1}, \cdots$.
  3. TC = the MTC of toric code. It has four simple objects $1, e, m, f$ and the following fusion rules:

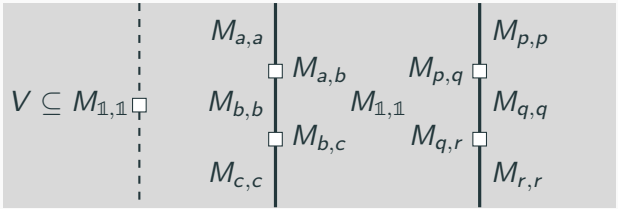
$$e \otimes e = m \otimes m = f \otimes f = 1, \quad m \otimes e = f.$$

- Three non-chiral symmetries (i.e. full field algebras Huang-K.:math/0511328):
  1. $P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1} \in \mathsf{Is} \boxtimes \overline{\mathsf{Is}} = \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_P = |\chi_0|^2$;
  2. $Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_Q = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2$
  3. $R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma \in \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_R = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2$

  We have $P \lneq Q \lneq R$.

- P, Q, R are condensable algebras in $\mathfrak{Z}_1(\mathsf{Is})$ and R is a Lagrangian algebra.
  1. $\phi_P : \mathrm{Mod}_P = (\mathfrak{Z}_1(\mathsf{Is}))^0_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$: condensing P gives $\mathfrak{Z}_1(\mathsf{Is})$;
  2. $\phi_Q : \mathrm{Mod}_Q = (\mathfrak{Z}_1(\mathsf{Is}))^0_Q \xrightarrow{\simeq} \mathsf{TC}$: condensing Q in $\mathfrak{Z}_1(\mathsf{Is})$ gives toric code TC Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334;
  3. $\phi_R : \mathrm{Mod}_R = (\mathfrak{Z}_1(\mathsf{Is}))^0_R \xrightarrow{\simeq} \mathrm{Vec}$: condensing R in $\mathfrak{Z}_1(\mathsf{Is})$ gives the trivial phase.

$$P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1}$$
$$Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi$$
$$R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma$$
$$\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$$
$$\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathsf{TC}$$
$$\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$$



- Four anomaly-free 1+1D gapless quantum liquids defined by triples: i.e. its 2+1D bulk topological order is trivial: $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$ and $\mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S}) \simeq \mathrm{Vec}$.
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathsf{Is})$: in this case $V = P \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2))$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  3. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2})$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})$: in this case $V = R = M_{\mathbb{1},\mathbb{1}}$.

  In all 4 cases, the space of non-chiral fields living on each 2-cells (i.e. $M_{\mathbb{1},\mathbb{1}}$) is given by the same modular-invariant closed CFT R.

- Gappable gapless edges of 2+1D toric code: $\mathrm{TC} = \mathfrak{Z}_1({}^{\mathfrak{B}}\mathcal{S})$, Chen-Jian-K.-You-Zheng:1903.12334, K.-Zheng:1912.01760
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})\mathcal{S}})$, where $\mathcal{S} = (\mathfrak{Z}_1(\mathsf{Is}))_Q$ is the fusion category of the right Q-modules in $\mathfrak{Z}_1(\mathsf{Is})$. $\mathcal{S}$ has 6 simple objects $\mathbb{1}, e, m, f, \chi_\pm$, where $\mathbb{1}, e, m, f$ can be identified with 4 anyons in the bulk and $\chi_\pm$ can be identified with two twist defects in the bulk.
  2. $(Q, \phi_Q, {}^{\mathrm{TC}}\mathrm{TC}) =$ the canoncial gapless edge;
  3. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Rep}(\mathbb{Z}_2))$, where $\phi : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$. Moreover, we have
  $$(R, \phi_R, \mathrm{Rep}(\mathbb{Z}_2)) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Rep}(\mathbb{Z}_2))}_{\text{the smooth gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}).}_{\text{an anomaly-free 2D gapless liquid}}$$
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}_{\mathbb{Z}_2})$:
  $$(R, \phi_R, \mathrm{Vec}_{\mathbb{Z}_2}) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Vec}_{\mathbb{Z}_2})}_{\text{the rough gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}$$
  5. (smooth/rough gapped edge) $\boxtimes$ (any anomaly-free 2D gapless liquid).